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# The Yangian symmetry in the spin Calogero model and its applications

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**Abstract.** By using the non-symmetric Hermite polynomials and a technique based on the Yangian Gelfand–Zetlin bases, we decompose the space of states of the Calogero model with spin into irreducible Yangian modules, construct an orthogonal basis of eigenvectors and derive product-type formulae for norms of these eigenvectors.

## 1. Introduction

The Calogero model and the Calogero–Sutherland model describe integrable quantum many-body systems with long-ranged interaction [7, 20]. In the Calogero model the particles move on the line $\mathbb{R}^1$, and in the Calogero–Sutherland model the particles move on the circle $S^1$. The spin Calogero–Sutherland model has a non-Abelian symmetry identified with the Yangian $Y(\mathfrak{gl}_n)$ [6]. In [21], the space of states of the spin Calogero–Sutherland model is decomposed into irreducible Yangian submodules and an orthogonal basis of eigenvectors is constructed. Product-type formulae for the norms of these eigenvectors are derived by using the irreducible decomposition of the space of states as the Yangian module and the technique based on the Yangian Gelfand–Zetlin bases [8, 17, 18].

On the other hand, similarities of the algebraic structures between the spin Calogero–Sutherland model and the spin Calogero model are pointed out in [13, 22]. In this paper they are summarized in section 4. Moreover, the spin Calogero model has a Yangian symmetry [5]. In this paper we introduce the Yangian symmetry in the spin Calogero model, in a little different way from [5], decompose the space of states of the spin Calogero model into irreducible Yangian submodules, and by using this decomposition construct an orthogonal basis of eigenvectors and derive product-type formulae for the norms of these eigenvectors similar to the spin Calogero–Sutherland model.

The similarity between these two models appears in the expression of the eigenbasis. The eigenbasis of the Calogero–Sutherland model is written in terms of the non-symmetric Jack polynomials, and the one in the Calogero model is written in terms of the non-symmetric Hermite polynomials. These two families of polynomials are deeply connected [3, 4], in particular see relation (4.21) of this paper. This is the key point why we can treat the Calogero model in a way similar to the one applied to the Calogero–Sutherland model in [21].

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## 2. Spin Calogero model with harmonic potential

Here we will define the Hamiltonian of the Calogero model and the space of states.

Introduce the Calogero Hamiltonian with spin as follows [7]

$$H_{\mathrm{CH}} := -h^2 H_{\mathrm{C}} + \omega^2 \sum_{j=1}^{N} x_j^2 \qquad H_{\mathrm{C}} := \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \sum_{1\leqslant j\neq k\leqslant N} \frac{\lambda^2 - \kappa\lambda P_{j,k}}{(x_j - x_k)^2}. \tag{2.1}$$

The Hamiltonian $H_{\mathrm{CH}}$ acts on wavefunctions of the form $\phi(x_1, \ldots, x_N|s_1, \ldots, s_N)$, where $s_i (1 \leqslant i \leqslant N)$ takes values in $\{1, \ldots, n\}$ and is called the spin variable.

The operator $P_{i,j}$ acts on the wavefunctions and changes the $i$th spin and $j$th spin:

$$(P_{i,j}\phi)(\ldots|\ldots, s_i, \ldots, s_j, \ldots) := \phi(\ldots|\ldots, s_j, \ldots, s_i, \ldots). \tag{2.2}$$

The constant $\kappa$ is $\pm 1$ and the system is called bosonic (resp. fermionic) if $\kappa = 1$ (resp. $= -1$). The wavefunction of the bosonic (resp. fermionic) Calogero model is symmetric (resp. antisymmetric) with respect to the exchangement of the coordinate and the spin at the same time.

*Remark.* For the spinless bosonic Calogero model, the wavefunction of the ground state is as follows

$$\tilde{\phi}_0 = \mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^{N} x_j^2} \prod_{j<k} |x_j - x_k|^{\lambda}. \tag{2.3}$$

We introduce the following gauge transformation and a new Hamiltonian,

$$\phi_0 := \prod_{j<k} |x_j - x_k|^{\lambda} \tag{2.4}$$

$$\tilde{H}_{\mathrm{C}} := \phi_0^{-1} H_{\mathrm{C}} \phi_0 = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{1\leqslant j\neq k\leqslant N} \left\{ \frac{2\lambda}{x_j - x_k} \frac{\partial}{\partial x_j} + \frac{\lambda(\kappa P_{i,j} - 1)}{(x_j - x_k)^2} \right\} \tag{2.5}$$

$$\tilde{H}_{\mathrm{CH}} := \phi_0^{-1} H_{\mathrm{CH}} \phi_0 = -h^2 \tilde{H}_{\mathrm{C}} + \omega^2 \sum_{j=1}^{N} x_j^2. \tag{2.6}$$

The spaces of states of the gauge transformed bosonic ($\kappa = 1$) (resp. fermionic ($\kappa = -1$)) Calogero model are

$$\mathcal{H}^{(\kappa)} := \bigcap_{i=1}^{N-1} \mathrm{Ker}(K_{i,i+1} P_{i,i+1} - \kappa 1) \subset \mathcal{H}(:= \mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^{N} x_j^2} \cdot \mathbb{C}[x_1, x_2, \ldots, x_N] \otimes (\otimes^N \mathbb{C}^n)). \tag{2.7}$$

Here $K_{i,j}$ exchanges $x_i$ and $x_j$, and $P_{i,j}$ exchanges the $i$th and $j$th component of $\otimes^N \mathbb{C}^n$.

We will define the scalar product on the space $\mathcal{H}^{(\kappa)}$. For this purpose we first define the scalar product on $\otimes^N \mathbb{C}^n$ and $\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^{N} x_j^2} \cdot \mathbb{C}[x_1, x_2, \ldots, x_N]$.

We fix the base $\{v_\epsilon\}_{\epsilon=1,\ldots,n}$ in $\mathbb{C}^n$ and define in $\otimes^N \mathbb{C}^n$ the Hermitian (sesquilinear) scalar product $\langle\cdot,\cdot\rangle_s$ by requiring pure tensors to be orthonormal:

$$\langle v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \cdots \otimes v_{\epsilon_N}, v_{\tau_1} \otimes v_{\tau_2} \otimes \cdots \otimes v_{\tau_N} \rangle_s := \prod_{i=1}^{N} \delta_{\epsilon_i,\tau_i} \qquad (\epsilon_i, \tau_i = 1, 2, \ldots, n). \tag{2.8}$$

In $\mathrm{e}^{-\frac{\omega}{2\hbar}\sum_{j=1}^{N}x_j^2}\cdot\mathbb{C}[x_1,x_2,\ldots,x_N](\ni f,g)$ define the scalar product $\langle\cdot,\cdot\rangle_c$ as follows.

$$\langle f,g\rangle_c := \left(\prod_{i=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}x_i\right)\prod_{j<k}|x_j-x_k|^{2\lambda}\overline{f(x_1,x_2,\ldots,x_N)}g(x_1,x_2,\ldots,x_N). \tag{2.9}$$

The Hermitian scalar product $\langle\cdot,\cdot\rangle$ in the space $\mathcal{H}$ is defined as the composition of the scalar products (2.8) and (2.9). For $f,g\in\mathrm{e}^{-\frac{\omega}{2\hbar}\sum_{j=1}^{N}x_j^2}\cdot\mathbb{C}[x_1,x_2,\ldots,x_N]$ and $u,v\in\otimes^N\mathbb{C}^n$ put

$$\langle f\otimes u, g\otimes v\rangle := \langle f,g\rangle_c\langle u,v\rangle_s \tag{2.10}$$

and extend the $\langle\cdot,\cdot\rangle$ on the entire space $\mathcal{H}$ by requiring it to be sesquilinear. We define the scalar product $\langle\cdot,\cdot\rangle_{(\kappa)}$ on the space $\mathcal{H}^{(\kappa)}$ by restricting the scalar product $\langle\cdot,\cdot\rangle$ to the subspace $\mathcal{H}^{(\kappa)}$.

## 3. Degenerate affine Hecke algebra and the non-symmetric Jack polynomial

In this section we recall the definition of the non-symmetric Jack polynomial.

We define the operator $d_i (1\leqslant i\leqslant N)$ which acts on $\mathbb{C}[x_1,x_2,\ldots,x_N]$ as follows

$$d_i := \frac{\partial}{\partial x_i} + \lambda\sum_{j\neq i}\frac{1-K_{i,j}}{x_i-x_j}. \tag{3.1}$$

Then we can check following relations

$$[d_i,d_j]=0 \qquad K_{i,j}d_i = d_jK_{i,j}. \tag{3.2}$$

*Remark.* If $v\in\mathcal{H}^{(\kappa)}$ then we have

$$\left(\sum_{i=1}^{N}d_i^2\right)v = \tilde{H}_{\mathrm{C}}v. \tag{3.3}$$

We put $\alpha = 1/\lambda$ and set

$$\tilde{d}_i := \alpha x_i d_i + \sum_{j>i}K_{i,j} \qquad \hat{d}_i := \tilde{d}_i - N. \tag{3.4}$$

Then $\hat{d}_i$ are the Dunkl operators which appear in the Calogero–Sutherland model [11, 6], (see (2.9))

$$\hat{d}_i = \alpha\frac{\partial}{\partial x_i} - \mathrm{i} + \sum_{j>i}\frac{x_j}{x_j-x_i}(K_{i,j}-1) - \sum_{j<i}\frac{x_i}{x_i-x_j}(K_{i,j}-1). \tag{3.5}$$

Remark that the operators $\tilde{d}_i$ and $K_{i,i+1}$ satisfy the relations of the degenerate affine Hecke algebra

$$[\tilde{d}_i,\tilde{d}_j]=0 \tag{3.6}$$
$$K_{i,i+1}\tilde{d}_i - \tilde{d}_{i+1}K_{i,i+1} = 1 \tag{3.7}$$
$$[\tilde{d}_iK_{j,j+1}]=0 \qquad |i-j|>1. \tag{3.8}$$

The actions of $\tilde{d}_i$ on monomials are as follows.

$$\tilde{d}_i\cdot x_1^{n_1}x_2^{n_2}\ldots x_N^{n_N} = (\alpha n_i - i + N)x_1^{n_1}x_2^{n_2}\ldots x_N^{n_N}$$

$$
+\sum_{j<i}\begin{cases}\displaystyle\sum_{l=n_j+1}^{n_i}\left(x_i^l x_j^{n_i+n_j-l}\prod_{m\neq i,j}x_m^{n_m}\right) & (n_i>n_j)\\ 0 & (n_i=n_j)\\ \displaystyle\sum_{l=n_i+1}^{n_j}-\left(x_i^l x_j^{n_i+n_j-l}\prod_{m\neq i,j}x_m^{n_m}\right) & (n_i<n_j)\end{cases}
$$

$$
-\sum_{j>i}\begin{cases}\displaystyle\sum_{l=n_j}^{n_i-1}\left(x_i^l x_j^{n_i+n_j-l}\prod_{m\neq i,j}x_m^{n_m}\right) & (n_i>n_j)\\ 0 & (n_i=n_j)\\ \displaystyle\sum_{l=n_i}^{n_j-1}-\left(x_i^l x_j^{n_i+n_j-l}\prod_{m\neq i,j}x_m^{n_m}\right) & (n_i<n_j).\end{cases} \tag{3.9}
$$

Let $\mathcal{M}_N := \{(m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N | m_1 \geqslant m_2 \geqslant \ldots \geqslant m_N \geqslant 0\}$ be the set of partitions. For $\boldsymbol{m} \in \mathcal{M}_N$ we set

$$
S^{\boldsymbol{m}} := \{\sigma \in \mathfrak{S}_N | i < j \text{ and } m_{\sigma i} = m_{\sigma j} \Rightarrow \sigma(i) < \sigma(j)\}. \tag{3.10}
$$

We can identify the pair $(\boldsymbol{m}, \sigma)(\boldsymbol{m}:$ partition, $\sigma \in S^{\boldsymbol{m}})$ and $\boldsymbol{n}(\in \mathbb{Z}^N_{\geqslant 0})$ as follows

$$
(\boldsymbol{m}, \sigma) \Leftrightarrow \boldsymbol{n} = (m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(N)}). \tag{3.11}
$$

Then $\sigma(\in S^{\boldsymbol{m}})$ satisfy the following property:

$$
\text{for all } 1 \leqslant i \leqslant N \qquad \sigma(i) = \#\{j \leqslant i | m_{\sigma(j)} \geqslant m_{\sigma(i)}\} + \#\{j > i | m_{\sigma(j)} > m_{\sigma(i)}\}. \tag{3.12}
$$

In the set $S^{\boldsymbol{m}}$ introduce the total ordering by setting

$$
\begin{aligned}
&\sigma \succ \sigma' \qquad \text{iff the last non-zero element in}\\
&(m_{\sigma(1)} - m_{\sigma'(1)}, m_{\sigma(2)} - m_{\sigma'(2)}, \ldots, m_{\sigma(N)} - m_{\sigma'(N)}) \text{ is } < 0.
\end{aligned} \tag{3.13}
$$

Note that the identity in $\mathfrak{S}_N$ is the maximal element in $S^{\boldsymbol{m}}$ in this ordering. Then in the set of pairs $(\boldsymbol{m}, \sigma)$ $(\boldsymbol{m} \in \mathcal{M}_N, \sigma \in S^{\boldsymbol{m}})$ the partial ordering is defined by

$$
(\boldsymbol{m}, \sigma) > (\tilde{\boldsymbol{m}}, \tilde{\sigma}) \qquad \text{iff} \begin{cases} \boldsymbol{m} > \tilde{\boldsymbol{m}} & \text{or}\\ \boldsymbol{m} = \tilde{\boldsymbol{m}} & \sigma \succ \tilde{\sigma}\end{cases} \tag{3.14}
$$

where $\boldsymbol{m} > \tilde{\boldsymbol{m}}$ means that $\boldsymbol{m}$ is greater than $\tilde{\boldsymbol{m}}$ in the dominance (natural) ordering in $\mathcal{M}_N$,

$$
\begin{aligned}
&\boldsymbol{m} > \tilde{\boldsymbol{m}} \Leftrightarrow\\
&\boldsymbol{m} \neq \tilde{\boldsymbol{m}} \qquad |\boldsymbol{m}| := \sum_{i=1}^N m_i = |\tilde{\boldsymbol{m}}|\\
&\text{and} \qquad \sum_{i=1}^j m_i \geqslant \sum_{i=1}^j \tilde{m}_i \qquad (1 \leqslant \forall j \leqslant N-1).
\end{aligned} \tag{3.15}
$$

The eigenvectors $\Phi^{\boldsymbol{m}}_\sigma(x) \in \mathbb{C}[x_1, x_2, \ldots, x_N]$ of the Dunkl operators are labelled by the pairs $(\boldsymbol{m}, \sigma)$ $(\boldsymbol{m} \in \mathcal{M}_N, \sigma \in S^{\boldsymbol{m}})$ and satisfy the following properties:

$$
\Phi^{\boldsymbol{m}}_\sigma(z) = x_1^{m_{\sigma(1)}} x_2^{m_{\sigma(2)}} \cdots x_N^{m_{\sigma(N)}} + \sum_{(\tilde{\boldsymbol{m}},\tilde{\sigma})<(\boldsymbol{m},\sigma)} c_{(\boldsymbol{m},\sigma);(\tilde{\boldsymbol{m}},\tilde{\sigma})} x_1^{\tilde{m}_{\tilde{\sigma}(1)}} x_2^{\tilde{m}_{\tilde{\sigma}(2)}} \cdots x_N^{\tilde{m}_{\tilde{\sigma}(N)}} \tag{3.16}
$$

$$
\begin{aligned}
&\tilde{d}_i \Phi^{\boldsymbol{m}}_\sigma(x) = \xi^{\boldsymbol{m}}_i(\sigma) \Phi^{\boldsymbol{m}}_\sigma(x) \qquad \text{where } \xi^{\boldsymbol{m}}_i(\sigma) := \alpha m_{\sigma(i)} + N - \sigma(i)\\
&(i = 1, 2, \ldots, N)
\end{aligned} \tag{3.17}
$$

$$
K_{i,i+1} \Phi^{\boldsymbol{m}}_\sigma(x) = \mathcal{A}^{\boldsymbol{m}}_i(\sigma) \Phi^{\boldsymbol{m}}_\sigma(x) + \mathcal{B}^{\boldsymbol{m}}_i(\sigma) \Phi^{\boldsymbol{m}}_{\sigma(i,i+1)}(x) \tag{3.18}
$$

where

$$\mathcal{A}_i^{\boldsymbol{m}}(\sigma) := \frac{1}{\xi_i^{\boldsymbol{m}}(\sigma) - \xi_{i+1}^{\boldsymbol{m}}(\sigma)}$$
$$\mathcal{B}_i^{\boldsymbol{m}}(\sigma) := \begin{cases} \dfrac{(\xi_i^{\boldsymbol{m}}(\sigma) - \xi_{i+1}^{\boldsymbol{m}}(\sigma))^2 - 1}{(\xi_i^{\boldsymbol{m}}(\sigma) - \xi_{i+1}^{\boldsymbol{m}}(\sigma))^2} & (m_{\sigma(i)} > m_{\sigma(i+1)}) \\ 0 & (m_{\sigma(i)} = m_{\sigma(i+1)}) \\ 1 & (m_{\sigma(i)} < m_{\sigma(i+1)}). \end{cases} \tag{3.19}$$

Note that for $\sigma \in S^{\boldsymbol{m}}$ we have $\sigma(i+1) = \sigma(i) + 1$ whenever $m_{\sigma(i)} = m_{\sigma(i+1)}$, and hence in this case,

$$K_{i,i+1}\Phi_\sigma^{\boldsymbol{m}}(x) = \Phi_\sigma^{\boldsymbol{m}}(x) \qquad (m_{\sigma(i)} = m_{\sigma(i+1)}). \tag{3.20}$$

We remark that if $\alpha > 0$ then the joint eigenvalues are all distinct and $\Phi_\sigma^{\boldsymbol{m}}(x)$ are well defined. We call these polynomials $\Phi_\sigma^{\boldsymbol{m}}(x)$ the non-symmetric Jack polynomials [9, 15]. If we set $\psi := x_N K_{N-1,N} K_{N-2,N-1} \ldots K_{1,2}$, we obtain (cf [14])

$$\psi \cdot \Phi^{\boldsymbol{n}}(x) = \Phi^{\boldsymbol{n}'}(x). \tag{3.21}$$

Here, $\Phi^{(m_{\sigma(1)},\ldots,m_{\sigma(N)})} := \Phi_\sigma^{\boldsymbol{m}}(x)$, $\boldsymbol{n}(\in \mathbb{Z}_{\geqslant 0}^N) := (n_1, n_2, \ldots n_N)$, and $\boldsymbol{n}' := (n_2, n_3, \ldots n_N, n_1 + 1)$.

For $f(x_1, x_2, \ldots, x_N)$, $g(x_1, x_2, \ldots, x_N) \in \mathbb{C}[x_1, x_2, \ldots, x_N]$ set

$$\langle f, g\rangle_J := \frac{1}{N!}\left(\prod_{i=1}^N \oint_{|w_i|=1} \frac{\mathrm{d}w_i}{2\pi\sqrt{-1}w_i}\right)\left(\prod_{i\neq j} 1 - \frac{w_i}{w_j}\right)^{\frac{1}{\alpha}} \times \overline{f(w_1, w_2, \ldots, w_N)}g(w_1, w_2, \ldots, w_N) \tag{3.22}$$

where the integration over each of the complex variables $w_i$ is taken along the unit circle in the complex plane.

If we denote by $A^\dagger$ the adjoint of the operator $A$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{J}}$, we obtain

$$K_{i,i+1}^\dagger = K_{i,i+1} \qquad x_i^\dagger = x_i^{-1} \tag{3.23}$$
$$\tilde{d}_i^\dagger = \tilde{d}_i \qquad \psi^\dagger = \psi^{-1}. \tag{3.24}$$

The non-symmetric Jack polynomials are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathrm{J}}$ and the recursive relations for the norms are

$$(1 - \mathcal{A}_i^{\boldsymbol{m}}(\sigma)^2)\langle\Phi_\sigma^{\boldsymbol{m}}(x), \Phi_\sigma^{\boldsymbol{m}}(x)\rangle_{\mathrm{J}} = \mathcal{B}_i^{\boldsymbol{m}}(\sigma)^2\langle\Phi_{\sigma(i,i+1)}^{\boldsymbol{m}}(x), \Phi_{\sigma(i,i+1)}^{\boldsymbol{m}}(x)\rangle_{\mathrm{J}} \tag{3.25}$$
$$\langle\Phi^{\boldsymbol{n}}(x), \Phi^{\boldsymbol{n}}(x)\rangle_{\mathrm{J}} = \langle\Phi^{\boldsymbol{n}'}(x), \Phi^{\boldsymbol{n}'}(x)\rangle_{\mathrm{J}}. \tag{3.26}$$

where $\boldsymbol{n}(\in \mathbb{Z}_{\geqslant 0}^N) := (n_1, n_2, \ldots n_N)$, and $\boldsymbol{n}' := (n_2, n_3, \ldots n_N, n_1 + 1)$.

## 4. Creation operators and the non-symmetric generalized Hermite polynomials

We will introduce the creation (annihilation) operators, the Dunkl operators for the Calogero model, and the non-symmetric generalized Hermite polynomials. These polynomials are also introduced in [3, 4] in a way a little different.

In this section, we will deal with the following scalar product (2.9):

$$\langle f, g\rangle_c := \left(\prod_{i=1}^N \int_{-\infty}^{\infty} \mathrm{d}x_i\right)\prod_{j<k}|x_j - x_k|^{2\lambda}\overline{f(x_1, x_2, \ldots, x_N)}g(x_1, x_2, \ldots, x_N).$$

First we can check that

$$d_i^\dagger = -d_i \qquad x_i^\dagger = x_i. \tag{4.1}$$

We define the annihilation operators as follows

$$A_i := hd_i + \omega x_i \qquad \bar{A}_i := \frac{1}{2h\omega} A_i. \tag{4.2}$$

Then the adjoints of the operators $A_i$ are

$$A_i^\dagger := -hd_i + \omega x_i. \tag{4.3}$$

We call $A_i^\dagger$ the creation operators. The commutation relations are

$$[A_i^\dagger, A_j^\dagger] = [\bar{A}_i, \bar{A}_j] = 0 \tag{4.4}$$

$$[A_i^\dagger, K_{j,k}] = [\bar{A}_i, K_{j,k}] = 0 \qquad (i \neq j, k) \tag{4.5}$$

$$A_i^\dagger K_{i,j} = K_{i,j} A_j^\dagger \qquad \bar{A}_i K_{i,j} = K_{i,j} \bar{A}_j \tag{4.6}$$

$$[\bar{A}_i, A_j^\dagger] = \delta_{i,j}\left(1 + \lambda \sum_{k\neq i} K_{i,k}\right) + (1-\delta_{i,j})\lambda K_{i,j}. \tag{4.7}$$

*Remark.* If we replace $A_i^\dagger \to x_i$, $\bar{A}_i \to d_i$, the same relations as (4.4)–(4.7) are satisfied [13, 22].

We introduce the Dunkl operators

$$\tilde{\Delta}_i := \alpha A_i^\dagger \bar{A}_i + \sum_{j>i} K_{i,j} \qquad \left(\alpha = \frac{1}{\lambda}\right) \tag{4.8}$$

then the operators $\tilde{\Delta}_i$ and $K_{i,i+1}$ satisfy the relations of the degenerate affine Hecke algebra.

$$[\tilde{\Delta}_i, \tilde{\Delta}_j] = 0 \tag{4.9}$$

$$K_{i,i+1}\tilde{\Delta}_i - \tilde{\Delta}_{i+1}K_{i,i+1} = 1 \tag{4.10}$$

$$[\tilde{\Delta}_i, K_{j,j+1}] = 0 \qquad |i-j| > 1. \tag{4.11}$$

The sum of the Dunkl operators is essentially the same as the Hamiltonian $\tilde{H}_{\mathrm{CH}}$

$$\left(\sum_{i=1}^{N} \tilde{\Delta}_i\right)v = \left(\frac{-1}{2h\omega\lambda}\tilde{H}_{\mathrm{CH}} + \frac{N}{2\lambda}\right)v \qquad \text{if } v \in \mathcal{H}^{(\kappa)}. \tag{4.12}$$

Next we will calculate the joint eigenfunctions of the operators $\tilde{\Delta}_i$, we will use the following relations later

$$\bar{A}_i \mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2} = 0 \qquad K_{i,k}\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2} = \mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}. \tag{4.13}$$

If we move $\bar{A}_i$ and $K_{i,j}$ to the right by relations (4.5)–(4.7) and use relations (4.13), we obtain

$$\begin{aligned}
\tilde{\Delta}_i (A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}\ldots(A_N^\dagger)^{n_N}\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2} &= (\alpha n_i - i + N)(A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}\ldots(A_N^\dagger)^{n_N}\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2} \\
&+ \sum_{j<i}\left\{\begin{array}{ll} \sum_{l=n_j+1}^{n_i}(A_i^\dagger)^l(A_j^\dagger)^{n_i+n_j-l}\prod_{m\neq i,j}(A_m^\dagger)_m^{n_m} & (n_i > n_j) \\ 0 & (n_i = n_j) \\ \sum_{l=n_i+1}^{n_j} -(A_i^\dagger)^l(A_j^\dagger)^{n_i+n_j-l}\prod_{m\neq i,j}(A_m^\dagger)^{n_m} & (n_i < n_j) \end{array}\right\}\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2} \\
&- \sum_{j>i}\left\{\begin{array}{ll} \sum_{l=n_j}^{n_i-1}(A_i^\dagger)^l(A_j^\dagger)^{n_i+n_j-l}\prod_{m\neq i,j}(A_m^\dagger)^{n_m} & (n_i > n_j) \\ 0 & (n_i = n_j) \\ \sum_{l=n_i}^{n_j-1} -(A_i^\dagger)^l(A_j^\dagger)^{n_i+n_j-l}\prod_{m\neq i,j}(A_m^\dagger)^{n_m} & (n_i < n_j) \end{array}\right\}\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}.
\end{aligned} \tag{4.14}$$

Comparing this with relation (3.9), the coefficients of $x_1^{k_1}\ldots x_N^{k_N}$ and $(A_1^\dagger)^{k_1}\ldots(A_N^\dagger)^{k_N}$ $e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}$ are the same for all $(k_1,\ldots,k_N)$. So the joint eigenfunction of the operators $\tilde{\Delta}_i$ are written by using the non- symmetric Jack polynomials:

$$\tilde{\Delta}_i\Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}=\xi_i^{\boldsymbol{m}}(\sigma)\Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2} \tag{4.15}$$

where $\xi_i^{\boldsymbol{m}}(\sigma):=\alpha m_{\sigma(i)}+N-\sigma(i)$ $(i=1,2,\ldots,N)$. Because of the relations

$$[A_i^\dagger,A_j^\dagger]=[x_i,x_j]=0 \tag{4.16}$$

$$[A_i^\dagger,K_{j,k}]=[x_i,K_{j,k}]=0 \qquad (i\neq j,k) \tag{4.17}$$

$$A_i^\dagger K_{i,j}=K_{i,j}A_j^\dagger \qquad x_iK_{i,j}=K_{i,j}x_j \tag{4.18}$$

we obtain

$$\begin{aligned}K_{i,i+1}\Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}&=\mathcal{A}_i^{\boldsymbol{m}}(\sigma)\Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}\\&\quad+\mathcal{B}_i^{\boldsymbol{m}}(\sigma)\Phi_{\sigma(i,i+1)}^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}\end{aligned} \tag{4.19}$$

$$\begin{aligned}\tilde{\psi}\cdot\Phi^{\boldsymbol{n}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}&:=A_N^\dagger K_{N-1,N}K_{N-2,N-1}\ldots K_{1,2}\Phi^{\boldsymbol{n}}(A_1^\dagger,\ldots,A_N^\dagger)\\&\quad\times e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}=\Phi^{\boldsymbol{n}'}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}.\end{aligned} \tag{4.20}$$

Here $(\boldsymbol{m}\in\mathcal{M}_N,\sigma\in S^{\boldsymbol{m}})$, $\boldsymbol{n}(\in\mathbb{Z}_{\geqslant 0}^N)=(n_1,n_2,\ldots,n_N)$, and $\boldsymbol{n}'=(n_2,n_3,\ldots,n_N,n_1+1)$. The coefficients $\mathcal{A}_i^{\boldsymbol{m}}(\sigma),\mathcal{B}_i^{\boldsymbol{m}}(\sigma)$ are the same as those in the Jack case (3.19).

We define the non-symmetric generalized Hermite polynomials as follows

$$\Phi_\sigma^{\boldsymbol{m}(H)}(x_1,\ldots,x_N):=e^{\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}\left(\frac{1}{2\omega}\right)^{|\boldsymbol{m}|}\Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}. \tag{4.21}$$

If we set the degrees as follows

$$\deg(\omega):=-2 \qquad \deg(\hbar):=0 \qquad \deg(x_i):=1 \tag{4.22}$$

the polynomial $\Phi_\sigma^{\boldsymbol{m}(H)}(x_1,\ldots,x_N)$ is homogeneous and $\deg(\Phi_\sigma^{\boldsymbol{m}(H)}(x_1,\ldots,x_N))=|\boldsymbol{m}|$. The non-symmetric generalized Hermite polynomials have the following expansion:

$$\Phi_\sigma^{\boldsymbol{m}(H)}(x)=\Phi_\sigma^{\boldsymbol{m}}(x)+\sum_{|\boldsymbol{n}|<|\boldsymbol{m}|}c_{\boldsymbol{n}}^{\boldsymbol{m},\sigma}x_1^{n_1}x_2^{n_2}\ldots x_N^{n_N}. \tag{4.23}$$

We note the relationship between the symmetric Jack (resp. Hermite) polynomials and the non-symmetric Jack (resp. Hermite) polynomials. We define the symmetric Jack (resp. Hermite) polynomials as the joint eigenfunctions for the operators $\sum_{i=1}^N\tilde{d}_k^i$ (resp. $e^{\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}\cdot\sum_{i=1}^N(\hat{\Delta}_k^i)\cdot e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}$) $(k=1,\ldots,N)$. They are labelled by the partitions and normalized so that the coefficient of the highest term (for the dominance ordering of the partitions) is 1 [19, 2]. The symmetric Jack (resp. Hermite) polynomials are written by the sum of the non-symmetric Jack (resp. Hermite) polynomials as follows

$$J^{\boldsymbol{m}}(x)=\sum_{\sigma\in S^{\boldsymbol{m}}}c_\sigma\Phi_\sigma^{\boldsymbol{m}}(x) \qquad \left(\text{resp. } H^{\boldsymbol{m}}(x)=\sum_{\sigma\in S^{\boldsymbol{m}}}c_\sigma\Phi_\sigma^{\boldsymbol{m}(H)}(x)\right). \tag{4.24}$$

Remark that the coefficients $c_\sigma$ for the Jack case and for the Hermite case are the same.

We calculate the recursion relations for the norms of $\Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}$. From relation (4.19) we obtain

$$\begin{aligned}&(1-\mathcal{A}_i^{\boldsymbol{m}}(\sigma)^2)\langle\Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2},\Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}\rangle_c\\&\quad=\mathcal{B}_i^{\boldsymbol{m}}(\sigma)^2\langle\Phi_{\sigma(i,i+1)}^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2},\Phi_{\sigma(i,i+1)}^{\boldsymbol{m}}(A_1^\dagger,\ldots,A_N^\dagger)e^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}\rangle_c.\end{aligned} \tag{4.25}$$

If we put $\boldsymbol{m}$ : partition ($\sigma \in S^{\boldsymbol{m}}$), $\boldsymbol{n} = (m_{\sigma(1)}, \ldots, m_{\sigma(N)})$, and $\boldsymbol{n}' = (n_2, n_3, \ldots n_N, n_1+1)$, by using relation (4.20) we obtain

$$\begin{aligned}
&\langle \Phi^{\boldsymbol{n}'}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, \Phi^{\boldsymbol{n}'}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c = \cdots \\
&\quad = \frac{2h\omega}{\alpha}\langle \Phi^{\boldsymbol{n}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, (\tilde{\Delta}_1 + \alpha)\Phi^{\boldsymbol{n}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c \\
&\quad = 2h\omega\left(m_{\sigma(1)} + 1 + \frac{N - \sigma(1)}{\alpha}\right) \\
&\qquad \times \langle \Phi^{\boldsymbol{n}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, \Phi^{\boldsymbol{n}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c.
\end{aligned} \tag{4.26}$$

By comparing it with the Jack cases (3.25), (3.26), we have

$$\frac{\langle \Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, \Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c}{\langle \Phi_{\sigma(i,i+1)}^{\boldsymbol{m}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, \Phi_{\sigma(i,i+1)}^{\boldsymbol{m}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c} = \frac{\langle \Phi_\sigma^{\boldsymbol{m}}(x), \Phi_\sigma^{\boldsymbol{m}}(x)\rangle_{\mathrm{J}}}{\langle \Phi_{\sigma(i,i+1)}^{\boldsymbol{m}}(x), \Phi_{\sigma(i,i+1)}^{\boldsymbol{m}}(x)\rangle_{\mathrm{J}}} \tag{4.27}$$

$$\frac{\langle \Phi^{\boldsymbol{n}'}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, \Phi^{\boldsymbol{n}'}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c}{\langle \Phi^{\boldsymbol{n}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, \Phi^{\boldsymbol{n}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c} = 2h\omega\left(m_{\sigma(1)} + 1 + \frac{N - \sigma(1)}{\alpha}\right)\frac{\langle \Phi^{\boldsymbol{n}'}(x), \Phi^{\boldsymbol{n}'}(x)\rangle_{\mathrm{J}}}{\langle \Phi^{\boldsymbol{n}}(x), \Phi^{\boldsymbol{n}}(x)\rangle_{\mathrm{J}}}. \tag{4.28}$$

Because of relations (4.24), (4.27), (4.28), we can calculate the norms of the symmetric Hermite polynomials

$$\begin{aligned}
&\langle H^{\boldsymbol{m}}(x)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, H^{\boldsymbol{m}}(x)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c = \left(\frac{\omega}{2h}\right)^{|\boldsymbol{m}|}\prod_{(i,j)\in\boldsymbol{m}}\left(j + \frac{N-i}{\alpha}\right) \\
&\qquad \times \langle J^{\boldsymbol{m}}(x), J^{\boldsymbol{m}}(x)\rangle_{\mathrm{J}}\frac{\langle \mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, \mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c}{\langle 1, 1\rangle_{\mathrm{J}}} \\
&\quad = \left(\frac{\omega}{2h}\right)^{|\boldsymbol{m}| + \frac{N(N-1)}{2\alpha}}\left(\frac{\omega\pi}{h}\right)^{\frac{N}{2}} N!\prod_{i=1}^N \Gamma\left(m_i + 1 + \frac{N-i}{\alpha}\right) \\
&\qquad \cdot \prod_{1\leqslant i<j\leqslant N}\frac{\Gamma\left(m_i - m_j + \frac{-i+j+1}{\alpha}\right)\Gamma\left(m_i - m_j + 1 + \frac{-i+j-1}{\alpha}\right)}{\Gamma\left(m_i - m_j + 1 + \frac{-i+j}{\alpha}\right)\Gamma\left(m_i - m_j + \frac{-i+j}{\alpha}\right)}.
\end{aligned} \tag{4.29}$$

$\prod_{(i,j)\in\boldsymbol{m}}$ means the product over the $(i, j)$-boxes contained in the Young diagram $\boldsymbol{m}$. To obtain formula (4.29), we used the following formulae [16, ch VI, 10.38, 2 proposition 3.7]

$$\langle J^{\boldsymbol{m}}(x), J^{\boldsymbol{m}}(x)\rangle_{\mathrm{J}} = \prod_{1\leqslant i<j\leqslant N}\frac{\Gamma\left(m_i - m_j + \frac{-i+j+1}{\alpha}\right)\Gamma\left(m_i - m_j + 1 + \frac{-i+j-1}{\alpha}\right)}{\Gamma\left(m_i - m_j + 1 + \frac{-i+j}{\alpha}\right)\Gamma\left(m_i - m_j + \frac{-i+j}{\alpha}\right)} \tag{4.30}$$

$$\left(\prod_{i=1}^N\int_{-\infty}^{\infty}\mathrm{e}^{-x_i^2}\,\mathrm{d}x_i\right)\prod_{j<k}|x_j - x_k|^{\frac{2}{\alpha}} = 2^{-\frac{N(N-1)}{2\alpha}}\pi^{\frac{N}{2}}\prod_{j=1}^{N-1}\frac{\Gamma(1 + \frac{j+1}{\alpha})}{\Gamma(1 + \frac{1}{\alpha})} \tag{4.31}$$

$$\left(\prod_{i=1}^N\oint_{|w_i|=1}\frac{\mathrm{d}w_i}{2\pi\sqrt{-1}w_i}\right)\left(\prod_{i\neq j}1 - \frac{w_i}{w_j}\right)^{\frac{1}{\alpha}} = \frac{\Gamma(1 + \frac{N}{\alpha})}{\Gamma(1 + \frac{1}{\alpha})^N}. \tag{4.32}$$

Remark that if $\omega/h = 1$ then formula (4.29) is written in [2, proposition 3.7].

## 5. Yangian $Y(\mathfrak{gl}_n)$ and the Yangian Gelfand–Zetlin bases

In this section we summarize the properties of the Yangian $Y(\mathfrak{gl}_n)$ which are used in this paper. The main attention is given to the Gelfand–Zetlin algebra and the canonical Yangian Gelfand–Zetlin bases in certain irreducible Yangian modules.

The Yangian $Y(\mathfrak{gl}_n)$ is a unital associative algebra generated by the elements 1 and $T^{(s)}_{a,b}$ where $a, b = 1, \ldots, n$ and $s = 1, 2, \ldots$ that are subject to the following relations:

$$[T^{(r)}_{a,b}, T^{(s+1)}_{c,d}] - [T^{(r+1)}_{a,b}, T^{(s)}_{c,d}] = T^{(r)}_{c,b}T^{(s)}_{a,d} - T^{(s)}_{c,b}T^{(r)}_{a,d} \qquad (r, s = 0, 1, 2, \ldots) \tag{5.1}$$

where $T^{(0)}_{a,b} := \delta_{a,b}1$.

Introducing the formal Taylor series in $u^{-1}$

$$T_{a,b}(u) = \delta_{a,b} + T^{(1)}_{a,b}u^{-1} + T^{(2)}_{a,b}u^{-2} + \cdots. \tag{5.2}$$

Define $\overset{k}{T}(u) (k = 1, 2)$ as follows

$$\overset{k}{T}(u) = \sum_{a,b=1}^{n} E^{(k)}_{a,b} \otimes T_{a,b}(u) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n) \otimes Y(\mathfrak{gl}_n[[u^{-1}]]. \tag{5.3}$$

Here $E^{(k)}_{a,b}$ are the standard matrix units that are acting in the $k$th tensor factor $\mathbb{C}^n$. If we put

$$R(u, v) = 1 + \frac{1}{u - v} \sum_{a,b=1}^{n} E^{(1)}_{a,b} \otimes E^{(2)}_{b,a} \tag{5.4}$$

then the defining relations of $Y(\mathfrak{gl}_n)$ are

$$R(u, v) \overset{1}{T}(u) \overset{2}{T}(v) = \overset{2}{T}(v) \overset{1}{T}(u) R(u, v). \tag{5.5}$$

Let $\boldsymbol{i} = (i_1, \ldots, i_m)$ and $\boldsymbol{j} = (j_1, \ldots, j_m)$ be two sequences of indices such that

$$1 \leqslant i_1 < \cdots < i_m \leqslant n \qquad \text{and} \qquad 1 \leqslant j_1 < \cdots < j_m \leqslant n. \tag{5.6}$$

Let $\mathfrak{S}_m$ be the symmetric group of degree $m$. Define

$$Q_{\boldsymbol{ij}}(u) = \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}(\sigma) \cdot T_{i_1, j_{\sigma(1)}}(u) T_{i_2, j_{\sigma(2)}}(u - 1) \ldots T_{i_m, j_{\sigma(m)}}(u - m + 1) \tag{5.7}$$

and

$$A_0(u) = 1 \qquad A_m(u) = Q_{\boldsymbol{ii}}(u) \qquad (m = 1, \ldots, n) \tag{5.8}$$

$$B_m(u) = Q_{\boldsymbol{ij}}(u) \qquad C_m(u) = Q_{\boldsymbol{ji}}(u) \qquad (m = 1, \ldots, n - 1) \tag{5.9}$$

where $\boldsymbol{i} = (1, \ldots, m)$ and $\boldsymbol{j} = (1, \ldots, m - 1, m + 1)$. The following propositions can be found in [17].

*Proposition 1 [17].* (a) The coefficients of $A_n(u)$ belong to the centre of the algebra $Y(\mathfrak{gl}_n)$.

(b) All the coefficients of $A_1(u), \ldots, A_n(u)$ pairwise commute.

By proposition 1, the coefficients $A^{(s)}_m$ of the series $A_1(u), \ldots, A_n(u)$:

$$A_m(u) = \sum_{s \geqslant 0} u^{-s} A^{(s)}_m \qquad (m = 1, 2, \ldots, n) \tag{5.10}$$

generate the commutative subalgebra in $Y(\mathfrak{gl}_n)$. This algebra is called Gelfand–Zetlin algebra and is denoted by $A(\mathfrak{gl}_n)$.

Let $V$ be an irreducible finite-dimensional $\mathfrak{gl}_n$-module and $E_{a,b}$ be the generators of $\mathfrak{gl}_n$. Denote by $v_\lambda$ the highest weight vector in $V$:

$$E_{a,a} \cdot v_\lambda = \lambda_a v_\lambda \qquad E_{a,b} \cdot v_\lambda = 0 \qquad a < b. \tag{5.11}$$

Then each difference $\lambda_a - \lambda_{a+1}$ is a non-negative integer. We assume that each $\lambda_a$ is also an integer. Denote by $\mathcal{T}_\lambda$ the set of all arrays $\Lambda$ with integral entries of the form

$$\begin{array}{c} \lambda_{n,1}\lambda_{n,2}\cdots\cdots\cdots\cdots\lambda_{n,n} \\ \lambda_{n-1,1}\cdots\cdots\lambda_{n-1,n-1} \\ \ddots \quad \cdots\cdots \\ \lambda_{2,1}\lambda_{2,2} \\ \lambda_{1,1} \end{array} \tag{5.12}$$

where $\lambda_{n,i} = \lambda_i$ and $\lambda_i \geqslant \lambda_{m,i}$ for all $i$ and $m$. The array $\Lambda$ is called a Gelfand–Zetlin scheme if

$$\lambda_{m,i} \geqslant \lambda_{m-1,i} \geqslant \lambda_{m,i+1} \tag{5.13}$$

for all possible $m$ and $i$. Denote by $\mathcal{S}_\lambda$ the subset in $\mathcal{T}_\lambda$ consisting of the Gelfand–Zetlin schemes [12].

Let us recall some facts about representations of the Yangian $Y(\mathfrak{gl}_n)$.

If we set $u' = u + h$, $v' = v + h (h \in \mathbb{C})$, relations (5.5) are also satisfied for $(u', v')$. Thus the map

$$T_{a,b}(u) \mapsto T_{a,b}(u + h) \tag{5.14}$$

defines an automorphism of the algebra $Y(\mathfrak{gl}_n)$. So if there is a representation $V$ of $Y(\mathfrak{gl}_n)$, we can construct another representation of $Y(\mathfrak{gl}_n)$ by the pullback through this automorphism.

We can regard the representation of the Lie algebra $\mathfrak{gl}_n$ as the representation of $Y(\mathfrak{gl}_n)$. This transpires due to the existence of the homomorphism $\pi_n$ from $Y(\mathfrak{gl}_n)$ to $U(\mathfrak{gl}_n)$: the universal enveloping algebra of $\mathfrak{gl}_n$:

$$\pi_n : T_{a,b}(u) \mapsto \delta_{a,b} + E_{b,a} u^{-1}. \tag{5.15}$$

Let $V_\lambda$ be the irreducible $\mathfrak{gl}_n$-module whose highest weight is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. We denote by $V_\lambda(h)$ the $Y(\mathfrak{gl}_n)$-module obtained from $V_\lambda$ by the pullback through this homomorphism and automorphism (5.15).

The Yangian $Y(\mathfrak{gl}_n)$ has the coproduct $\Delta : Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n) \otimes Y(\mathfrak{gl}_n)$. It is given as follows

$$\Delta(T_{a,b}(u)) = \sum_{c=1}^{n} T_{a,c}(u) \otimes T_{c,b}(u). \tag{5.16}$$

So if there are representations $V_i (i = 1, \ldots, M)$ of the Yangian $Y(\mathfrak{gl}_n)$, we can construct the representation $V_1 \otimes V_2 \otimes \cdots \otimes V_M$ of $Y(\mathfrak{gl}_n)$:

$$\begin{aligned} T_{a,b}(u) \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_M) &= \Delta^{(M)} \circ \cdots \circ \Delta^{(2)}(T_{a,b}(u))(v_1 \otimes v_2 \otimes \ldots \otimes v_M) \\ &= \sum_{k_1 \ldots k_{M-1}} T_{a,k_1}(u) v_1 \otimes T_{k_1,k_2}(u) v_2 \otimes \cdots \otimes T_{k_{M-1},b}(u) v_M. \end{aligned} \tag{5.17}$$

Hereafter we consider the following representation of the Yangian $Y(\mathfrak{gl}_n)$:

$$W = V_{\lambda^{(1)}}(h^{(1)}) \otimes V_{\lambda^{(2)}}(h^{(2)}) \otimes \cdots \otimes V_{\lambda^{(M)}}(h^{(M)}) \tag{5.18}$$

where we assume that $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ for all $r \neq s$.

Let us set $\rho_0(u) = 1$ and for $m = 1, \ldots, n$ let us define

$$\rho_m(u) = \prod_{s=1}^{M} \prod_{i=1}^{m} (u - i + 1 + h^{(s)}) \tag{5.19}$$

and

$$a_m(u) = \rho_m(u) A_m(u) \qquad m = 0, \ldots, n \tag{5.20}$$

$$b_m(u) = \rho_m(u) B_m(u) \qquad m = 1, \ldots, n-1 \tag{5.21}$$

$$c_m(u) = \rho_m(u) C_m(u) \qquad m = 1, \ldots, n-1. \tag{5.22}$$

Then $a_m(u)$, $b_m(u)$, and $c_m(u)$ are polynomials in $u$.

Let us fix a set of Gelfand–Zetlin schemes

$$\Lambda^{(s)} = (\lambda^{(s)}_{m,i} | 1 \leqslant i \leqslant m \leqslant n) \in \mathcal{T}_{\lambda^{(s)}} \qquad (s = 1, \ldots, M) \tag{5.23}$$

and define the following polynomials for $m = 0, \ldots, n$.

$$\varpi_{m,\Lambda^{(1)},\ldots,\Lambda^{(M)}}(u) = \prod_{s=1}^{M} \prod_{i=1}^{m} (u + \lambda^{(s)}_{m,i} - i + 1 + h^{(s)}). \tag{5.24}$$

Note that all the zeros of the $m$th polynomial

$$\nu^{(s)}_{m,i} = i - \lambda^{(s)}_{m,i} - 1 - h^{(s)} \tag{5.25}$$

are pairwise distinct due to our assumption on the parameters $h^{(1)}, \ldots, h^{(M)}$.

For the pairs $(m, m')(1 \leqslant m' \leqslant m \leqslant n)$, we introduce the ordering,

$$(m, m') \prec (l, l') \Leftrightarrow m' < l' \qquad \text{or} \qquad (m' = l' \text{ and } m > l). \tag{5.26}$$

Let $v_{\mathrm{hwv}} \in W$ be the vector, which is the tensor product of the highest weight vectors $v^{(s)}_{\mathrm{hwv}}$ of the Lie algebra $\mathfrak{gl}_n (s = 1, \ldots, M)$. Then consider the following vector in $W$

$$v_{\Lambda^{(1)},\ldots,\Lambda^{(M)}} = \prod_{(m,m')}^{\rightarrow} \left( \prod_{\substack{(s,t) \\ 1 \leqslant t \leqslant \lambda^{(s)}_{n,m'} - \lambda^{(s)}_{m,m'}}} b_m(\nu^{(s)}_{m,m'} - t) \right) \cdot v_{\mathrm{hwv}}. \tag{5.27}$$

Remark that for each fixed $m$ the elements $b_m(\nu^{(s)}_{m,m'} - t) \in \mathrm{End}(W)$ mutually commute.

Then the following propositions are satisfied (see [18, 21]).

*Proposition 2 [18].* For every $m = 1, \ldots, n$ we have the equality

$$a_m(u) \cdot v_{\Lambda^{(1)},\ldots,\Lambda^{(M)}} = \varpi_{m,\Lambda^{(1)},\ldots,\Lambda^{(M)}}(u) \cdot v_{\Lambda^{(1)},\ldots,\Lambda^{(M)}}. \tag{5.28}$$

*Proposition 3 [18].* $Y(\mathfrak{gl}_n)$-module $W$ is irreducible if $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ for all $r \neq s$.

*Proposition 4.* $v_{\Lambda^{(1)},\ldots,\Lambda^{(M)}} (\Lambda^{(r)} \in \mathcal{S}_{\lambda^{(r)}}$ for every $r \in \{1, \ldots, M\})$ form a base of $W$.

## 6. Yangian in the spin Calogero model

In this section we recall the definition of the Yangian action in the spin Calogero model and establish some properties of this action—in particular the self-adjointness of the operators giving the action of the Gelfand–Zetlin algebra.

For $\kappa = \pm$ define the Monodromy operator $\hat{T}^{(\kappa)}_0(u) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathcal{H})[[u^{-1}]]$ by

$$\hat{T}^{(\kappa)}_0(u) = \sum_{a,b=1}^{n} E_{a,b} \otimes \hat{T}^{(\kappa)}_{a,b}(u) := \left(1 + \frac{P_{0,1}}{u - \kappa \tilde{\Delta}_1}\right) \left(1 + \frac{P_{0,2}}{u - \kappa \tilde{\Delta}_2}\right) \cdots \left(1 + \frac{P_{0,N}}{u - \kappa \tilde{\Delta}_N}\right) \tag{6.1}$$

where $P_{0,i}$ in this definition is the permutation operator of the zeroth and $i$th tensor factors $\mathbb{C}^n$ in the tensor product

$$\underset{0}{\mathbb{C}}^n \otimes \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes \underset{1}{\mathbb{C}}^n \otimes \underset{2}{\mathbb{C}}^n \otimes \cdots \otimes \underset{N}{\mathbb{C}}^n = \underset{0}{\mathbb{C}}^n \otimes \mathcal{H}. \tag{6.2}$$

The $E_{a,b} \in \mathrm{End}(\mathbb{C}^n)$ is the standard matrix unit in the basis $\{v_\epsilon\}$. The operators $\hat{T}_{a,b}^{(\kappa),(s)} \in \mathrm{End}(\mathcal{H})$ obtained by expanding the Monodromy matrix $\hat{T}_{a,b}^{(\kappa)}(u)$:

$$\hat{T}_{a,b}^{(\kappa)}(u) = \delta_{a,b} 1 + \sum_{s \geqslant 1} u^{-s} \hat{T}_{a,b}^{(\kappa),(s)} \tag{6.3}$$

satisfy the defining relations (5.1) of the algebra $Y(\mathfrak{gl}_n)$. By using the relations of the degenerate affine Hecke algebra (4.9)–(4.11) we can show

$$(K_{i,i+1} - \kappa P_{i,i+1}) \hat{T}^{(\kappa)}(u) = \hat{T}^{(\kappa)}(u)|_{\tilde{\Delta}_i \leftrightarrow \tilde{\Delta}_{i+1}} (K_{i,i+1} - \kappa P_{i,i+1}). \tag{6.4}$$

Then the operators $\hat{T}_{a,b}^{(\kappa),(s)}$ leave the subspace $\mathcal{H}^{(\kappa)}$ invariant. We will set

$$T_{a,b}^{(\kappa)}(u) := \hat{T}_{a,b}^{(\kappa)}(u)|_{\mathcal{H}^{(\kappa)}} \in \mathrm{End}(\mathcal{H}^{(\kappa)})[[u^{-1}]] \qquad (a, b = 1, 2, \ldots, n). \tag{6.5}$$

Denote the generating series which give the action of the Gelfand–Zetlin algebra in the Yangian representation defined by the Monodromy matrix (6.1) by $A_1^{(\kappa)}(u), A_2^{(\kappa)}(u), \ldots, A_n^{(\kappa)}(u)$. The $A_n^{(\kappa)}(u)$ is just the quantum determinant of the $T_{a,b}^{(\kappa)}(u)$. Hence

$$[A_n^{(\kappa)}(u), T_{a,b}^{(\kappa)}(v)] = 0 \qquad (a, b = 1, 2, \ldots, n). \tag{6.6}$$

The explicit expression for the quantum determinant [6]:

$$A_n^{(\kappa)}(u) = \prod_{i=1}^{N} \frac{u + 1 - \kappa \tilde{\Delta}_i}{u - \kappa \tilde{\Delta}_i} \tag{6.7}$$

shows that the spin Calogero Hamiltonian (2.6) is an element in the centre of the Yangian action and hence is an element in the Gelfand–Zetlin algebra.

By using the self-adjointness $\tilde{\Delta}_i^\dagger = \tilde{\Delta}_i$, we can show the following propositions (cf [21] for the proofs).

*Proposition 5.*

$$T_{a,b}^{(\kappa)}(u)^\dagger = T_{b,a}^{(\kappa)}(u) \qquad (\kappa = -, +). \tag{6.8}$$

*Proposition 6.*

$$\begin{aligned} &A_m^{(\kappa)}(u)^\dagger = A_m^{(\kappa)}(u) \qquad B_m^{(\kappa)}(u)^\dagger = C_m^{(\kappa)}(u) \\ &C_m^{(\kappa)}(u)^\dagger = B_m^{(\kappa)}(u) \qquad (\kappa = -, +). \end{aligned} \tag{6.9}$$

## 7. Decomposition of the space of states into irreducible Yangian submodules

In this section we construct the decomposition of the space of states of the spin Calogero model into irreducible submodules of the Yangian action. The contents of this section is almost the same as section 5 of [21].

### *7.1. Irreducible decomposition of the space of states with respect to the Yangian action. Fermionic case*

In this section we describe the decomposition of the space of states in the fermionic spin Calogero model: $\mathcal{H}^{(-)}$ into irreducible subrepresentations with respect to the $Y(\mathfrak{gl}_n)$-action (6.1) ($\kappa = -$). Let $E^{\boldsymbol{m}} := \oplus_{\sigma \in S^{\boldsymbol{m}}} \mathbb{C}\Phi^{\boldsymbol{m}}_\sigma(A^\dagger_1, \ldots, A^\dagger_N)\mathrm{e}^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}$ ($\boldsymbol{m} \in \mathcal{M}_N$), and let

$$F^{\boldsymbol{m}} := (E^{\boldsymbol{m}} \otimes (\otimes^N \mathbb{C}^n)) \cap \mathcal{H}^{(-)}. \tag{7.1}$$

Then (6.5) implies that the space $F^{\boldsymbol{m}}$ is invariant with respect to the Yangian action defined by (6.1) with $\kappa = -$. Since $\Phi^{\boldsymbol{m}}_\sigma(A^\dagger_1, \ldots, A^\dagger_N)\mathrm{e}^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}$ ($\boldsymbol{m} \in \mathcal{M}_N, \sigma \in S^{\boldsymbol{m}}$) form a base in $\mathbb{C}[x_1, x_2, \ldots, x_N]$ we have

$$\mathcal{H}^{(-)} = \bigoplus_{\boldsymbol{m} \in \mathcal{M}_N} F^{\boldsymbol{m}}. \tag{7.2}$$

To describe each of the components $F^{\boldsymbol{m}}$ explicitly we need to make several definitions.

Let $W^{\boldsymbol{m}}_{(-)} \subset \otimes^N \mathbb{C}^n (\boldsymbol{m} \in \mathcal{M}_N)$ be defined by

$$W^{\boldsymbol{m}}_{(-)} := \bigcap_{1 \leqslant i \leqslant N \mathrm{st} m_i = m_{i+1}} \mathrm{Ker}(P_{i,i+1} + 1). \tag{7.3}$$

Note that $\dim W^{\boldsymbol{m}}_{(-)} = 0$ unless $\boldsymbol{m} \in \mathcal{M}^{(n)}_N$ where

$$\mathcal{M}^{(n)}_N := \{\boldsymbol{m} \in \mathcal{M}_N | \#\{m_k | m_k = i\} \leqslant n (i \in \mathbb{Z})\}. \tag{7.4}$$

For $p \in \{1, 2, \ldots, n\}$ let $\lambda$ be the highest weight of the fundamental $\mathfrak{gl}_n$-module:

$$\lambda = (\underbrace{1, 1, \ldots, 1}_{p}, \underbrace{0, 0, \ldots, 0}_{n-p}) \qquad (1 \leqslant p \leqslant n). \tag{7.5}$$

For a highest weight of this form and $h \in \mathbb{C}$ denote the corresponding $Y(\mathfrak{gl}_n)$-module $V_\lambda(h)$ by $V_p(h)$. As a linear space the $V_p(h)$ is realized as the totally antisymmetrized tensor product of $\mathbb{C}^n$:

$$V_p(h) = \cap_{i=1}^{p-1} \mathrm{Ker}(P_{i,i+1} + 1) \qquad \subset \otimes^p \mathbb{C}^n \qquad (1 \leqslant p \leqslant n) \tag{7.6}$$

with normalization chosen so that the $\mathfrak{gl}_n$ highest weight vector in $V_p(h)$ is

$$\omega_p := \sum_{\sigma \in \mathfrak{S}_p} (-1)^{l(\sigma)} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(p)}. \tag{7.7}$$

For an $\boldsymbol{m} \in \mathcal{M}^{(n)}_N$ let $M$ be the number of distinct elements in the sequence

$$\boldsymbol{m} = (m_1, m_2, \ldots, m_N)$$

and let $p_s (1 \leqslant p_s \leqslant n, s = 1, 2, \ldots, M)$ be the multiplicities of the elements in the $\boldsymbol{m}$:

$$\begin{aligned} m_1 = m_2 = \cdots = m_{p_1} > m_{1+p_1} = m_{2+p_1} = \cdots = m_{p_2+p_1} > \cdots > m_{1+p_{M-1}+\cdots+p_2+p_1} \\ = m_{2+p_{M-1}+\cdots+p_2+p_1} = \cdots = m_{p_M+\cdots+p_2+p_1 \equiv N}. \end{aligned} \tag{7.8}$$

With $\xi^{\boldsymbol{m}}_i := \xi^{\boldsymbol{m}}_i(\mathrm{id})$ (3.17) set

$$h^{(s)}_{\boldsymbol{m}} := \xi^{\boldsymbol{m}}_{1+p_1+p_2+\cdots+p_{s-1}} \qquad (p_0 := 0, s = 1, 2, \ldots, M). \tag{7.9}$$

Then for the linear space $W^{\boldsymbol{m}}_{(-)}$ (7.3) we have

$$W^{\boldsymbol{m}}_{(-)} = \begin{cases} V_{p_1}(h^{(1)}_{\boldsymbol{m}}) \otimes \cdots \otimes V_{p_M}(h^{(M)}_{\boldsymbol{m}}) \subset \otimes^N \mathbb{C}^n & \text{when } \boldsymbol{m} \in \mathcal{M}^{(n)}_N \\ \emptyset & \text{when } \boldsymbol{m} \notin \mathcal{M}^{(n)}_N. \end{cases} \tag{7.10}$$

When $\boldsymbol{m} \in \mathcal{M}_N^{(n)}$ the $W_{(-)}^{\boldsymbol{m}}$ is the Yangian module with the Yangian action defined by the coproduct (5.17).

For any $\sigma \in S^{\boldsymbol{m}}$ (3.10) define $\check{\mathbb{R}}^{(-)}(\sigma) \in \mathrm{End}(\otimes^N \mathbb{C}^n)$ by the following recursion relation:

$$\check{\mathbb{R}}^{(-)}(\mathrm{id}) := 1 \tag{7.11}$$

$$\check{\mathbb{R}}^{(-)}(\sigma(i, i+1)) := -\check{R}_{i,i+1}(\xi_i^{\boldsymbol{m}}(\sigma) - \xi_{i+1}^{\boldsymbol{m}}(\sigma))\check{\mathbb{R}}^{(-)}(\sigma) \qquad (m_{\sigma(i)} > m_{\sigma(i+1)}) \tag{7.12}$$

where the $R$-matrix is given by

$$\check{R}_{i,i+1}(u) := u^{-1} + P_{i,i+1}. \tag{7.13}$$

Due to property (3.10) of the set $S^{\boldsymbol{m}}$ this recursion relation is sufficient to define $\check{\mathbb{R}}^{(-)}(\sigma)$ for all $\sigma \in S^{\boldsymbol{m}}$. The definition of the $\check{\mathbb{R}}^{(-)}(\sigma)$ is unambiguous by virtue of the Yang–Baxter equation satisfied by the $R$-matrix (7.13).

For $\boldsymbol{m} \in \mathcal{M}_N$ define the map $U_{(-)}^{\boldsymbol{m}} : \otimes^N \mathbb{C}^n \to \mathcal{H}$ by setting for $v \in \otimes^N \mathbb{C}^n$

$$U_{(-)}^{\boldsymbol{m}} v := \sum_{\sigma \in S^{\boldsymbol{m}}} \Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2} \otimes \check{\mathbb{R}}^{(-)}(\sigma)v. \tag{7.14}$$

*Theorem 7.* For any $\boldsymbol{m} \in \mathcal{M}_N$ we have

$$U_{(-)}^{\boldsymbol{m}} : W_{(-)}^{\boldsymbol{m}} \to F^{\boldsymbol{m}} \tag{7.15}$$

and the $U_{(-)}^{\boldsymbol{m}}$ is an isomorphism of the $Y(\mathfrak{gl}_n)$-modules $W_{(-)}^{\boldsymbol{m}}$ and $F^{\boldsymbol{m}}$.

The proof of this theorem is almost the same as the one given in appendix A of [21]. (Exchange $\Phi_\sigma^{\boldsymbol{m}}(z) \leftrightarrow \Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}$.)

For now let us note that from this theorem it follows that the Yangian highest weight vector $\Omega_{\boldsymbol{m}}^{(-)}$ in $F^{\boldsymbol{m}}$ is given by

$$\Omega_{\boldsymbol{m}}^{(-)} = U_{(-)}^{\boldsymbol{m}}\omega_{\boldsymbol{m}} = \sum_{\sigma \in S^{\boldsymbol{m}}} \Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2} \otimes \check{\mathbb{R}}^{(-)}(\sigma)\omega_{\boldsymbol{m}} \tag{7.16}$$

where the $\omega_{\boldsymbol{m}}$ is the highest weight vector in $W_{(-)}^{\boldsymbol{m}}$:

$$\omega_{\boldsymbol{m}} := \omega_{p_1} \otimes \omega_{p_2} \otimes \cdots \otimes \omega_{p_M}. \tag{7.17}$$

From corollary 3.9 in [18] it follows that the modules $F^{\boldsymbol{m}}$ are irreducible if $\alpha \notin \mathbb{Q}$ since in this case in (7.10) we have $h_{\boldsymbol{m}}^{(s)} - h_{\boldsymbol{m}}^{(r)} \notin \mathbb{Z}$ when $s \neq r$. Using the results of [1] for the Yangian version, we can verify that the $F^{\boldsymbol{m}}$ are irreducible under the weaker condition: $\alpha \in \mathbb{R} \setminus \mathbb{Q}_{\leqslant 0}$.

### *7.2. Irreducible decomposition of the space of states with respect to the Yangian action. Bosonic case*

The decomposition of the space of states of the bosonic spin Calogero model: $\mathcal{H}^{(+)}$ into irreducible sub-representations with respect to the $Y(\mathfrak{gl}_n)$-action (6.1) ($\kappa = +$) is carried out along the same lines as the one for the fermionic case.

Let for $\boldsymbol{m} \in \mathcal{M}_N$ the $E^{\boldsymbol{m}}$ be defined as in section 7.1, and let

$$B^{\boldsymbol{m}} := (E^{\boldsymbol{m}} \otimes (\otimes^N \mathbb{C}^n)) \cap \mathcal{H}^{(+)}. \tag{7.18}$$

Then (6.5) implies that the space $B^{\boldsymbol{m}}$ is invariant with respect to the Yangian action defined by (6.1) with $\kappa = +$. Since $\Phi_\sigma^{\boldsymbol{m}}(A_1^\dagger, \ldots, A_N^\dagger)\mathrm{e}^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}$ ($\boldsymbol{m} \in \mathcal{M}_N, \sigma \in S^{\boldsymbol{m}}$) form a base in $\mathbb{C}[x_1, x_2, \ldots, x_N]$ we have

$$\mathcal{H}^{(+)} = \bigoplus_{\boldsymbol{m} \in \mathcal{M}_N} B^{\boldsymbol{m}}. \tag{7.19}$$

To describe each of the components $B^{\boldsymbol{m}}$ explicitly we make several definitions analogous to those made in section 7.1.

Let $W^{\boldsymbol{m}}_{(+)} \subset \otimes^N \mathbb{C}^n$ $(\boldsymbol{m} \in \mathcal{M}_N)$ be defined by

$$W^{\boldsymbol{m}}_{(+)} := \bigcap_{1 \leqslant i \leqslant N \, \mathrm{st} \, m_i = m_{i+1}} \mathrm{Ker}(P_{i,i+1} - 1). \tag{7.20}$$

For $p = 1, 2, \ldots$ let $\lambda$ be the following $\mathfrak{gl}_n$ highest weight:

$$\lambda = (p, \underbrace{0, 0, \ldots, 0}_{n-1}). \tag{7.21}$$

For a highest weight of this form and $h \in \mathbb{C}$ denote the corresponding $Y(\mathfrak{gl}_n)$-module $V_\lambda(h)$ by $V^p(h)$. As a linear space the $V^p(h)$ is realized as the totally symmetrized tensor product of $\mathbb{C}^n$:

$$V^p(h) = \cap_{i=1}^{p-1} \mathrm{Ker}(P_{i,i+1} - 1) \qquad \subset \otimes^p \mathbb{C}^n \qquad (p = 1, 2, \ldots). \tag{7.22}$$

We choose normalization so that the highest weight vector in $V_p(h)$ is equal to $v_1^{\otimes p}$. As in the fermionic case, for an $\boldsymbol{m} \in \mathcal{M}_N$ let $M$ be the number of distinct elements in the sequence $\boldsymbol{m} = (m_1, m_2, \ldots, m_N)$, and let $p_s (s = 1, 2, \ldots, M)$ be the multiplicities of the elements in the $\boldsymbol{m}$:

$$\begin{aligned} m_1 = m_2 = \cdots = m_{p_1} > m_{1+p_1} = m_{2+p_1} = \cdots = m_{p_2+p_1} > \cdots > m_{1+p_{M-1}+\cdots+p_2+p_1} \\ = m_{2+p_{M-1}+\cdots+p_2+p_1} = \cdots = m_{p_M+\cdots+p_2+p_1 \equiv N}. \end{aligned} \tag{7.23}$$

With $\xi_i^{\boldsymbol{m}} := \xi_i^{\boldsymbol{m}}(\mathrm{id})$ (3.17) set

$$h^{(s)}_{\boldsymbol{m}} := -\xi^{\boldsymbol{m}}_{1+p_1+p_2+\cdots+p_{s-1}} \qquad (p_0 := 0, s = 1, 2, \ldots, M). \tag{7.24}$$

Then for the linear space $W^{\boldsymbol{m}}_{(+)}$ (7.20) we have

$$W^{\boldsymbol{m}}_{(+)} = V^{p_1}(h^{(1)}_{\boldsymbol{m}}) \otimes V^{p_2}(h^{(2)}_{\boldsymbol{m}}) \otimes \cdots \otimes V^{p_M}(h^{(M)}_{\boldsymbol{m}}) \subset \otimes^N \mathbb{C}^n. \tag{7.25}$$

The $W^{\boldsymbol{m}}_{(+)}$ is the Yangian module with the Yangian action defined by the coproduct (5.17).

For any $\sigma \in S^{\boldsymbol{m}}$ (3.10) define $\check{\mathbb{R}}^{(+)}(\sigma) \in \mathrm{End}(\otimes^N \mathbb{C}^n)$ by the following recursion relation:

$$\check{\mathbb{R}}^{(+)}(\mathrm{id}) := 1 \tag{7.26}$$

$$\check{\mathbb{R}}^{(+)}(\sigma(i, i+1)) := \check{R}_{i,i+1}(-\xi_i^{\boldsymbol{m}}(\sigma) + \xi_{i+1}^{\boldsymbol{m}}(\sigma))\check{\mathbb{R}}^{(+)}(\sigma) \qquad (m_{\sigma(i)} > m_{\sigma(i+1)}) \tag{7.27}$$

where the $R$-matrix $\check{R}_{i,i+1}(u)$ is given by (7.13).

As in the fermionic case, due to property (3.10) of the set $S^{\boldsymbol{m}}$ this recursion relation is sufficient to define $\check{\mathbb{R}}^{(+)}(\sigma)$ for all $\sigma \in S^{\boldsymbol{m}}$. The definition of the $\check{\mathbb{R}}^{(+)}(\sigma)$ is unambiguous by virtue of the Yang–Baxter equation satisfied by the $R$-matrix (7.13).

For $\boldsymbol{m} \in \mathcal{M}_N$ define the map $U^{\boldsymbol{m}}_{(+)} : \otimes^N \mathbb{C}^n \to \mathcal{H}$ by setting for $v \in \otimes^N \mathbb{C}^n$

$$U^{\boldsymbol{m}}_{(+)} v := \sum_{\sigma \in S^{\boldsymbol{m}}} \Phi^{\boldsymbol{m}}_\sigma (A_1^\dagger, \ldots, A_N^\dagger) \mathrm{e}^{-\frac{\omega}{2\hbar} \sum_{j=1}^N x_j^2} \otimes \check{\mathbb{R}}^{(+)}(\sigma) v. \tag{7.28}$$

*Theorem 8.* For any $\boldsymbol{m} \in \mathcal{M}_N$ we have

$$U^{\boldsymbol{m}}_{(+)} : W^{\boldsymbol{m}}_{(+)} \to B^{\boldsymbol{m}} \tag{7.29}$$

where $U^{\boldsymbol{m}}_{(+)}$ is an isomorphism of the $Y(\mathfrak{gl}_n)$-modules $W^{\boldsymbol{m}}_{(+)}$ and $B^{\boldsymbol{m}}$.

We omit the proof of this theorem since it is a straightforward modification of the proof of the theorem 1 given in [21]. From this theorem it follows that the Yangian highest weight vector $\Omega^{(+)}_{\boldsymbol{m}}$ in $B^{\boldsymbol{m}}$ is given by

$$\Omega^{(+)}_{\boldsymbol{m}} = U^{\boldsymbol{m}}_{(+)} v_1^{\otimes N} = \sum_{\sigma \in S^{\boldsymbol{m}}} \Phi^{\boldsymbol{m}}_\sigma (A_1^\dagger, \ldots, A_N^\dagger) \mathrm{e}^{-\frac{\omega}{2\hbar} \sum_{j=1}^N x_j^2} \otimes \check{\mathbb{R}}^{(+)}(\sigma) v_1^{\otimes N}. \tag{7.30}$$

## 8. Norms of the highest weight vectors in the irreducible Yangian submodules

In this section we will show the concrete expression of the norms $\langle \Omega^{(+)}_{\boldsymbol{m}}, \Omega^{(+)}_{\boldsymbol{m}}\rangle_{(+)}$ and $\langle \Omega^{(-)}_{\boldsymbol{m}}, \Omega^{(-)}_{\boldsymbol{m}}\rangle_{(-)}$. The method of calculation is the same as [21] section 6. (In the calculation of the norms of the highest weight vectors, we will use the norms of the symmetric generalized Hermite polynomials $\langle H_{\boldsymbol{m}}(x)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, H_{\boldsymbol{m}}(x)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c$ instead of $\langle J^{\boldsymbol{m}}(z), J^{\boldsymbol{m}}(z)\rangle_{\mathrm{J}}$.)

*Proposition 9.*

(Bosonic cases) For $\boldsymbol{m} \in \mathcal{M}_N$ we have

$$\langle \Omega^{(+)}_{\boldsymbol{m}}, \Omega^{(+)}_{\boldsymbol{m}}\rangle_{(+)} = (2\omega)^{|\boldsymbol{m}|}\langle H^{\boldsymbol{m}}(x)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, H^{\boldsymbol{m}}(x)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c \tag{8.1}$$

where the norm $\langle H^{\boldsymbol{m}}(x)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}, H^{\boldsymbol{m}}(x)\mathrm{e}^{-\frac{\omega}{2h}\sum_{j=1}^N x_j^2}\rangle_c$ of the symmetric generalized Hermite polynomial is given by formula (4.29).

(Fermionic cases) For $\boldsymbol{m} \in \mathcal{M}^{(n)}_N$ we have

$$\begin{aligned}\langle \Omega^{(-)}_{\boldsymbol{m}}, \Omega^{(-)}_{\boldsymbol{m}}\rangle_{(-)} = &\prod_{1\leqslant s<t\leqslant M} \frac{\Gamma\left(\frac{h^{(s)}_{\boldsymbol{m}}-h^{(t)}_{\boldsymbol{m}}}{\alpha} + \frac{p_t}{\alpha} + 1\right)\Gamma\left(\frac{h^{(s)}_{\boldsymbol{m}}-h^{(t)}_{\boldsymbol{m}}}{\alpha} - \frac{p_s}{\alpha}\right)}{\Gamma\left(\frac{h^{(s)}_{\boldsymbol{m}}-h^{(t)}_{\boldsymbol{m}}}{\alpha} + \frac{p_t-p_s}{\alpha} + \theta(p_s \leqslant p_t)\right)\Gamma\left(\frac{h^{(s)}_{\boldsymbol{m}}-h^{(t)}_{\boldsymbol{m}}}{\alpha} + \theta(p_s > p_t)\right)}\\ &\times\prod_{i=1}^N \Gamma\left(m_i + 1 + \frac{N-i}{\alpha}\right)\cdot\prod_{s=1}^M \frac{\Gamma\left(\frac{p_s}{\alpha}+1\right)}{\left\{\Gamma\left(\frac{1}{\alpha}+1\right)\right\}^{p_s}}\\ &\cdot h^{-|\boldsymbol{m}|}\left(\frac{\omega}{2h}\right)^{\frac{N(N-1)}{2\alpha}}\left(\frac{\omega\pi}{h}\right)^{\frac{N}{2}} N!\end{aligned} \tag{8.2}$$

where

$$\theta(x) := \begin{cases} 1 & \text{when } x \text{ is true}\\ 0 & \text{when } x \text{ is false.}\end{cases} \tag{8.3}$$

## 9. Eigenbases of the Gelfand–Zetlin algebra in the irreducible Yangian submodules and norms of the eigenvectors

In this section we construct eigenbases of the operator-valued series $A^{(\kappa)}_1(u), A^{(\kappa)}_2(u), \ldots, A^{(\kappa)}_n(u)$ within each of the irreducible $Y(\mathfrak{gl}_n)$- submodules $F^{\boldsymbol{m}}$ ($\boldsymbol{m} \in \mathcal{M}^{(n)}_N$) ($\kappa = -1$—fermionic case) and $B^{\boldsymbol{m}}(\boldsymbol{m} \in \mathcal{M}_N)$ ($\kappa = 1$—bosonic case), and compute norms of the eigenvectors that form these eigenbases.

Due to the isomorphisms given by the theorems 7 and 8 the construction of the eigenbases is carried out by a straightforward application of the results of Nazarov and Tarasov [18].

Let us fix a partition $\boldsymbol{m} = (m_1, m_2, \ldots, m_N) \in \mathcal{M}_N$ and let for $\kappa = -1\boldsymbol{m} \in \mathcal{M}^{(n)}_N \subset \mathcal{M}_N$. Define the number $p_s$ to be the multiplicities of the elements in the $\boldsymbol{m}$ (see (7.8)). In the fermionic case $\boldsymbol{m} \in \mathcal{M}^{(n)}_N$ and $p_s \in \{1, 2, \ldots, n\}$ ($s = 1, 2, \ldots, M$).

For $p \in \{1, 2, \ldots, n\}$ let $\mathcal{S}^{(-)}_p$ denote the set of all Gelfand–Zetlin schemes $\Lambda$ that are associated with the irreducible $\mathfrak{gl}_n$-module with the highest weight (cf section 7)

$$(\underbrace{1, 1, \ldots, 1}_{p}, \underbrace{0, 0, \ldots, 0}_{n-p}). \tag{9.1}$$

An element of $\mathcal{S}_p^{(-)}$ is an array of the form

$$\begin{array}{c} \lambda_{n,1}\lambda_{n,2}\cdots\cdots\cdots\cdots\lambda_{n,n} \\ \lambda_{n-1,1}\cdots\cdots\lambda_{n-1,n-1} \\ \ddots \quad \cdots\cdots \\ \lambda_{2,1}\lambda_{2,2} \\ \lambda_{1,1} \end{array} \tag{9.2}$$

where

$$(\lambda_{m,1},\lambda_{m,2},\ldots,\lambda_{m,m}) = (\underbrace{1,1,\ldots,1}_{l_m},\underbrace{0,0,\ldots,0}_{m-l_m}) \qquad (m=1,2,\ldots,n) \qquad l_n = p \tag{9.3}$$

and either

$$l_m = l_{m+1} \qquad \text{or} \qquad l_m = l_{m+1}-1 \qquad (m=1,2,\ldots,n-1). \tag{9.4}$$

For $p \in \mathbb{N}$ let $\mathcal{S}_p^{(+)}$ denote the set of all Gelfand–Zetlin schemes $\Lambda$ that are associated with the irreducible $\mathfrak{gl}_n$-module with the highest weight (cf section 7)

$$(p,\underbrace{0,0,\ldots,0}_{n-1}). \tag{9.5}$$

An element of $\mathcal{S}_p^{(+)}$ is a Gelfand–Zetlin scheme of the form

$$\begin{array}{c} \alpha_n 0\cdots\cdots\cdots\cdots 0 \\ \alpha_{n-1}0\cdots\cdots 0 \\ \ddots \quad \cdots\cdots \\ \alpha_2 \; 0 \\ \alpha_1 \end{array} \tag{9.6}$$

where

$$\alpha_m \leqslant \alpha_{m+1} \qquad (m=1,2,\ldots,n-1) \qquad \alpha_n = p. \tag{9.7}$$

Now let us define the following operator-valued series for the bosonic case set

$$a_m^{(+)}(u) = A_m^{(+)}(u) \qquad b_m^{(+)}(u) = B_m^{(+)}(u) \qquad c_m^{(+)}(u) = C_m^{(+)}(u) \tag{9.8}$$

and for the fermionic case set

$$a_m^{(-)}(u) = \Delta(u)A_m^{(-)}(u) \qquad b_m^{(-)}(u) = \Delta(u)B_m^{(-)}(u) \qquad c_m^{(-)}(u) = \Delta(u)C_m^{(-)}(u)$$

where $\Delta(u) = \prod_{i=1}^N (u+\hat{\Delta}_i)$. Then from proposition 6 it follows that

$$a_m^{(\kappa)}(u)^\dagger = a_m^{(\kappa)}(u) \qquad b_m^{(\kappa)}(u)^\dagger = c_m^{(\kappa)}(u) \qquad c_m^{(\kappa)}(u)^\dagger = b_m^{(\kappa)}(u) \qquad \kappa = -,+. \tag{9.9}$$

For a collection of Gelfand–Zetlin schemes $\Lambda^{(1)},\ldots,\Lambda^{(M)}$ such that $\Lambda^{(s)} \in \mathcal{S}_{p_s}^{(\kappa)}$ $(s=1,2,\ldots,M)$ define the following vector (cf section 5):

$$v^{(\kappa)}_{\Lambda^{(1)},\ldots,\Lambda^{(M)}} = \overrightarrow{\prod_{(m,m')}}\left(\prod_{\substack{(s,t)\\ 1\leqslant t\leqslant \lambda^{(s)}_{n,m'}-\lambda^{(s)}_{m,m'}}} b_m^{(\kappa)}(\nu^{(s)}_{m,m'}-t)\right)\cdot\Omega^{(\kappa)}_{\boldsymbol{m}} \tag{9.10}$$

$$v^{(\kappa)}_{\Lambda^{(1)},\ldots,\Lambda^{(M)}} \in \begin{cases} F^{\boldsymbol{m}} & (\kappa=-) \\ B^{\boldsymbol{m}} & (\kappa=+). \end{cases} \tag{9.11}$$

Here

$$\nu^{(s)}_{m,m'} = m' - \lambda^{(s)}_{m,m'} - 1 - h^{(s)}_{\boldsymbol{m}} \tag{9.12}$$

and the $h_{\boldsymbol{m}}^{(s)}$ are defined by (7.9) (the fermionic case) and (7.24) (the bosonic cases). From proposition 4 and the theorems 7 and 8 it follows that the set

$$\{v^{(\kappa)}_{\Lambda^{(1)},\dots,\Lambda^{(M)}} | \Lambda^{(s)} \in \mathcal{S}^{(\kappa)}_{p_s} (s = 1, 2, \dots, M)\} \tag{9.13}$$

is a base of $F^{\boldsymbol{m}}$ (resp. $B^{\boldsymbol{m}}$) when $\kappa = -$ (resp. $+$). Due to proposition 5.28 this is an eigenbase of the operators generating the Gelfand–Zetlin algebra:

$$A^{(\kappa)}_m(u) v^{(\kappa)}_{\Lambda^{(1)},\dots,\Lambda^{(M)}} = \mathcal{A}^{(\kappa)}_m(u; \boldsymbol{m})_{\Lambda^{(1)},\dots,\Lambda^{(M)}} v^{(\kappa)}_{\Lambda^{(1)},\dots,\Lambda^{(M)}} \qquad (m = 1, 2, \dots, n) \tag{9.14}$$

where the eigenvalues are defined by

$$\mathcal{A}^{(-)}_m(u; \boldsymbol{m})_{\Lambda^{(1)},\dots,\Lambda^{(M)}} = \prod_{s=1}^{M} \frac{u + 1 + h^{(s)}_{\boldsymbol{m}}}{u + 1 + h^{(s)}_{\boldsymbol{m}} - l^{(s)}_m} \qquad (\Lambda^{(s)} \in \mathcal{S}^{(-)}_{p_s}) \tag{9.15}$$

$$\mathcal{A}^{(+)}_m(u; \boldsymbol{m})_{\Lambda^{(1)},\dots,\Lambda^{(M)}} = \prod_{s=1}^{M} \frac{u + h^{(s)}_{\boldsymbol{m}} + \alpha^{(s)}_m}{u + h^{(s)}_{\boldsymbol{m}}} \qquad (\Lambda^{(s)} \in \mathcal{S}^{(+)}_{p_s}). \tag{9.16}$$

Since $\langle \Phi^{\boldsymbol{m}}_\sigma(A^\dagger_1, \dots, A^\dagger_N) \mathrm{e}^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2}, \Phi^{\boldsymbol{n}}_\tau(A^\dagger_1, \dots, A^\dagger_N) \mathrm{e}^{-\frac{\omega}{2\hbar}\sum_{j=1}^N x_j^2} \rangle_c = 0$ when $\boldsymbol{m} \neq \boldsymbol{n}$, the subspaces $F^{\boldsymbol{m}}$ (resp. $B^{\boldsymbol{m}}$) are pairwise orthogonal.

For $\alpha > 0$ one can verify that the data $\boldsymbol{m} \in \mathcal{M}_N$, $(\Lambda^{(1)}, \Lambda^{(2)} \dots, \Lambda^{(M)})$ $(\Lambda^{(s)} \in \mathcal{S}^{(\kappa)}_{p_s})$ are uniquely restored from the collection of rational functions

$$\mathcal{A}^{(\kappa)}_1(u; \boldsymbol{m})_{\Lambda^{(1)},\dots,\Lambda^{(M)}}, \mathcal{A}^{(\kappa)}_2(u; \boldsymbol{m})_{\Lambda^{(1)},\dots,\Lambda^{(M)}}, \dots, \mathcal{A}^{(\kappa)}_n(u; \boldsymbol{m})_{\Lambda^{(1)},\dots,\Lambda^{(M)}}. \tag{9.17}$$

That is the joint spectrum of eigenvalues of the Gelfand–Zetlin algebra is simple. Since $A^{(\kappa)}_m(u)$ are self-adjoint, we obtain the following.

*Proposition 10.* For $\boldsymbol{m} \in \mathcal{M}^{(n)}_N$ (resp. $\boldsymbol{m} \in \mathcal{M}_N$) the set

$$\{v^{(\kappa)}_{\Lambda^{(1)},\dots,\Lambda^{(M)}} | \Lambda^{(s)} \in \mathcal{S}^{(\kappa)}_{p_s} (s = 1, 2, \dots, M)\} \tag{9.18}$$

with $\kappa = -$ (resp. $\kappa = +$) is an orthogonal base of $F^{\boldsymbol{m}}$ (resp. $B^{\boldsymbol{m}}$).

The norms of the eigenvectors $v^{(\kappa)}_{\Lambda^{(1)},\dots,\Lambda^{(M)}}$ are as follows

*Proposition 11.*

(Bosonic case) Let $\boldsymbol{m} \in \mathcal{M}_N$ and $\Lambda^{(s)} \in \mathcal{S}^{(+)}_{p_s}$ $(s = 1, 2, \dots, M)$. If we write a Gelfand–Zetlin scheme $\Lambda^{(s)}$ as in (9.6):

$$\Lambda^{(s)} = \begin{array}{l} \alpha^{(s)}_n 0 \cdots\cdots\cdots\cdots 0 \\ \quad \alpha^{(s)}_{n-1} 0 \cdots\cdots 0 \\ \qquad \ddots \ \ \cdots\cdots \\ \qquad\quad \alpha^{(s)}_2 0 \\ \qquad\quad \alpha^{(s)}_1 \end{array} \tag{9.19}$$

then the norm of the vector $v^{(+)}_{\Lambda^{(1)},\dots,\Lambda^{(M)}}$ is

$$\langle v^{(+)}_{\Lambda^{(1)},\dots,\Lambda^{(M)}}, v^{(+)}_{\Lambda^{(1)},\dots,\Lambda^{(M)}} \rangle_{(+)} = \langle \Omega^{(+)}_{\boldsymbol{m}}, \Omega^{(+)}_{\boldsymbol{m}} \rangle_{(+)}$$
$$\cdot \prod_{1 \leqslant m \leqslant n} \left\{ \prod_{1 \leqslant s \leqslant M} \frac{(\alpha^{(s)}_n - \alpha^{(s)}_m)!(\alpha^{(s)}_n - \alpha^{(s)}_{m-1})!(\alpha^{(s)}_m!)^2}{(\alpha^{(s)}_m - \alpha^{(s)}_{m-1})!(\alpha^{(s)}_n!)^2} \right.$$
$$\left. \times \left\{ \prod_{\substack{(s,s') \\ s \neq s'}} \prod_{a=\alpha^{(s)}_m}^{\alpha^{(s)}_n - 1} \frac{(-a + \alpha^{(s')}_n + h^{(s')}_{\boldsymbol{m}} - h^{(s)}_{\boldsymbol{m}})(-1 - a + \alpha^{(s')}_{m-1} + h^{(s')}_{\boldsymbol{m}} - h^{(s)}_{\boldsymbol{m}})}{(-1 - a + h^{(s')}_{\boldsymbol{m}} - h^{(s)}_{\boldsymbol{m}})^2} \right\} \right.$$

$$\times \prod_{\substack{(s,s')\\ s<s'}} \frac{(\alpha_n^{(s')} - \alpha_n^{(s)} + h_{\boldsymbol{m}}^{(s')} - h_{\boldsymbol{m}}^{(s)})}{(\alpha_m^{(s')} - \alpha_m^{(s)} + h_{\boldsymbol{m}}^{(s')} - h_{\boldsymbol{m}}^{(s)})}\Bigg\} \tag{9.20}$$

where the $h_{\boldsymbol{m}}^{(s)}$ are defined by (7.24) with $\kappa = +$.

(Fermionic case) Let $\boldsymbol{m} \in \mathcal{M}_N^{(n)}$ and $\Lambda^{(s)} \in \mathcal{S}_{p_s}^{(-)}$ $(s = 1, 2, \ldots, M)$. As in (9.2), define $l_m^{(s)}$ associated with the Gelfand–Zetlin scheme $\Lambda^{(s)}$ by the conditions $\lambda^{(s)}_{m,l_m^{(s)}} = 1$ and $\lambda^{(s)}_{m,l_m^{(s)}+1} = 0$. Then the norm of the vector $v^{(-)}_{\Lambda^{(1)},\ldots,\Lambda^{(M)}}$ is

$$\begin{aligned}
\langle v^{(-)}_{\Lambda^{(1)},\ldots,\Lambda^{(M)}}, v^{(-)}_{\Lambda^{(1)},\ldots,\Lambda^{(M)}}\rangle_{(-)} &= \langle \Omega^{(-)}_{\boldsymbol{m}}, \Omega^{(-)}_{\boldsymbol{m}}\rangle_{(-)} \\
&\times\Bigg\{ \prod_{1\leqslant s\leqslant M} \prod_{\substack{(m,m')\\ \lambda^{(s)}_{m,m'}\neq\lambda^{(s)}_{n,m'}}} (m'-1)!^2 (p_s+1-m')!^2 \Bigg\}\Bigg\{ \prod_{\substack{(s,s')\\ s<s'}} \prod_{\substack{(m,m')\\ \lambda^{(s)}_{m,m'}\neq\lambda^{(s)}_{n,m'} \\ \lambda^{(s')}_{m,m'}\neq\lambda^{(s')}_{n,m'}}} \\
&\times \frac{\prod_{j=0}^{p_s}(m'-j-1+h_{\boldsymbol{m}}^{(s')} - h_{\boldsymbol{m}}^{(s)})^2 \prod_{j=0}^{p_{s'}}(m'-j-1+h_{\boldsymbol{m}}^{(s)} - h_{\boldsymbol{m}}^{(s')})^2}{(h_{\boldsymbol{m}}^{(s)} - h_{\boldsymbol{m}}^{(s')})^4} \Bigg\} \\
&\times\Bigg\{ \prod_{\substack{(s,s')\\ s\neq s'}} \prod_{\substack{(m,m')\\ \lambda^{(s)}_{m,m'}\neq\lambda^{(s)}_{n,m'} \\ \lambda^{(s')}_{m,m'}=\lambda^{(s')}_{n,m'}}} \Bigg[ (m'-l_m^{(s')} + h_{\boldsymbol{m}}^{(s')} - h_{\boldsymbol{m}}^{(s)}) \prod_{j=0}^{p_s}(m'-j-1+h_{\boldsymbol{m}}^{(s')} - h_{\boldsymbol{m}}^{(s)})^2 \Bigg] \\
&\times[(m'-1-l_{m-1}^{(s')} + h_{\boldsymbol{m}}^{(s')} - h_{\boldsymbol{m}}^{(s)})(m'-1-l_m^{(s')} + h_{\boldsymbol{m}}^{(s')} - h_{\boldsymbol{m}}^{(s)}) \\
&\times(m'-l_{m+1}^{(s')} + h_{\boldsymbol{m}}^{(s')} - h_{\boldsymbol{m}}^{(s)})]^{-1}\Bigg\}
\end{aligned} \tag{9.21}$$

where $h_{\boldsymbol{m}}^{(s)}$ are defined by (7.9) with $\kappa = -$. In these product formulae the $s$ and $s'$ range from 1 to $M$ and $(m, m')$ $(n \geqslant m \geqslant m' \geqslant 1)$ are coordinates of points in a Gelfand–Zeltin scheme of $\mathfrak{gl}_n$.

The proof is the same as the proof of proposition 14 in [21]. (Also see appendix B of [21].)

Together with proposition 9, this proposition gives the norm formulae for the orthogonal eigenbasis of the spin Calogero model.

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